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On the calculation of the available potential energy of an isolated perturbation in a density-stratified fluid

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Two methods for calculating the available potential energy (APE) of an isolated feature in a density-stratified fluid, such as an internal solitary wave or an eddy, are compared. The first formulation calculates the APE by integrating the perturbation potential energy density E_w. The second uses an available potential energy density E_a. Both formulations are based on the reference density obtained by adiabatically rearranging the density field to a state of minimum energy. It is shown, under more general conditions than used previously, that (i) the integrals of E_w and E_a over a finite domain are identical; and (ii) that for an isolated feature in an unbounded domain, the far-field density $\bar{\rho}(z)$ can be used as the reference density if E_a is used to find the APE. This is not the case when E_w is used, hence use of the available potential energy density formulation is simpler in this situation.

1. Introduction

Available potential energy (APE) is the potential energy that can be converted to other forms of mechanical energy, e.g. kinetic energy, and which is ultimately available for mixing. Hence determining the APE is an important problem. Relatively simple cases include isolated disturbances such as internal solitary-like waves (Klymak & Moum 2003; Klymak *et al.* 2006; Scotti, Beardsley & Butman 2006; Lamb 2007) or eddies (Prater & Sanford 1994; Benzohra & Millot 1995). The available potential energy of more complicated flows, such as the interaction of eddies and convection (Legg & McWilliams 2001) and eddies interacting with a sloping thermocline (Henning & Vallis 2004) has also been considered, as has the available potential energy of the ocean (Huang 2005). It is an important concept for understanding mixing processes (Winters *et al.* 1995; Arneborg 2002).

Here two different means of calculating the available potential energy of an isolated feature are compared. For simplicity, we assume an incompressible, inviscid, non-diffusive, two-dimensional flow vertically confined to lie between two horizontal, rigid boundaries at $z = -H$ and $z = 0$. All the results are directly applicable to three-dimensional flows and, because the Boussinesq approximation is not made, can be easily extended to include flows with a free surface. The results are also valid for both continuous and layered stratifications.

The first method is based on calculating the difference in the potential energy of the perturbed state and of a reference state obtained by adiabiatically rearranging the density field to the state with minimum potential energy (Lorenz 1955; Hebert 1988; Winters *et al.* 1995). In the resulting stable stratification, $\bar{\rho}_r(z)$, the isopycnals lie along geopotentials. Thus, in this method the perturbation potential energy density

$$E_w = \rho_w g z, \tag{1.1}$$

where $\rho_w = \rho - \bar{\rho}_r(z)$ is the density perturbation, is integrated over the volume under consideration. The second method uses the available potential energy density,

$$E_a(x, z, t) = g \int_z^{z^*(x,z,t)} (\bar{\rho}(s) - \rho(x, z, t))\,\mathrm{d}s, \tag{1.2}$$

where $z^*(x, z, t)$ is the height of the fluid parcel at (x, z) at time t in the reference state. E_w and E_a have the same integrated values over the domain so either formulation can be used to compute the total available potential energy; however use of the available potential energy density E_a has several advantages, many of which have been pointed out by previous authors (Andrews 1981; Holliday & McIntyre 1981; Shepherd 1993; Scotti *et al.* 2006; Lamb 2007). These include the fact that E_a is positive definite, that it reduces to the traditional linear potential energy density $g^2\rho_w^2/2N^2$ in the small-amplitude limit, where N is the buoyancy frequency of the reference state, that it is independent of the coordinate system, and that, in combination with the kinetic energy density E_k, it satisfies a relatively simple conservation law. If the reference density is invertible, with inverse $\bar{z}_r(\rho)$ satisfying $z^*(x, z, t) = \bar{z}_r(\rho(x, z, t))$, an alternative expression for E_a is

$$E_a(x, z, t) = g \int_{\bar{\rho}(z)}^{\rho(x,z,t)} (z - \bar{z}_r(s))\,\mathrm{d}s \tag{1.3}$$

(Shepherd 1993; Scotti *et al.* 2006; Lamb 2007).

The perturbation and potential energy densities are related by

$$E_a = E_w + (\bar{\rho}_r(z) g z + \bar{p}_r(z)) - (\rho z^* g + \bar{p}_r(z^*)), \tag{1.4}$$

where $\bar{p}_r(z)$ is a reference pressure in hydrostatic balance with $\bar{\rho}_r(z)$. If the reference density is invertible we can define

$$f(\rho) = -\rho \bar{z}_r(\rho) g - \bar{p}_r(\bar{z}_r(\rho)), \tag{1.5}$$

and it follows that

$$E_a = E_w + f(\rho) - f(\bar{\rho}). \tag{1.6}$$

The main purpose of this note is to point out a further advantage, namely that when determining the available potential energy of an isolated feature in an infinite domain, the far-field density can be used as the reference density. Integrating E_a across the finite region occupied by the isolated feature gives the available potential energy. In contrast, when integrating E_w the far-field density cannot be used as the reference density. Instead, it is necessary to calculate the APE in a finite domain by sorting the density field and then taking the limit as the length of the domain goes to infinity (Hebert 1988). Hebert (1988) also considered a Boussinesq available potential energy density (Reid, Elliot & Olson 1981) which is a small-amplitude approximation to E_a. He showed that its accuracy is restricted to small-amplitude perturbations. Use of the available potential energy density E_a has no such restriction.

This paper is organized as follows. In §2 the theoretical foundations of the two formulations are presented. In §3 we prove that the two formulations are equivalent. This was demonstrated previously by Holliday & McIntyre (1981); however their proof assumed an infinitely differentiable reference density with a convergent Taylor

Series. The proof presented here is valid for discontinuous stratifications such as a two-layer fluid. Section 3 concludes with an example, consisting of a two-layer fluid with an isolated square-well perturbation, which illustrates some of the differences between the two formulations. In particular it is shown that the spatial distributions of E_w and E_a are different and that in order to compute the APE of the square-well depression in an infinite domain one can use the far-field density as the reference density in the expression for E_a and integrate over the isolated feature. It is also shown that this cannot be done when using E_w. The difference arises because the vertical integral of E_a is quadratic in the amplitude of the perturbation while the vertical integral of E_w has a term that is linear in amplitude. In §4 we extend this result by proving that for a general isolated feature the available potential energy in an infinite domain can be calculated by integrating the available potential energy density using the far-field density as the reference density state. Results are summarized in §5.

2. Available potential energy formulations

For simplicity we consider a two-dimensional domain. The results are equally valid for three-dimensional features as, for example, can be deduced by sorting the density field in each horizontal direction separately. The available potential energy in a closed domain $(x, z) \in [-L, L] \times [-H, 0]$ is the integral of the perturbation potential energy $E_w = \rho_w g z$:

$$APE_L = \int_{-L}^{L} \int_{-H}^{0} \rho_w g z \, \mathrm{d}z \, \mathrm{d}x, \tag{2.1}$$

where

$$\rho_w(x, z, t) = \rho(x, z, t) - \bar{\rho}_L(z). \tag{2.2}$$

Here $\rho(x, z, t)$ is the fluid density and the reference density $\bar{\rho}_L(z)$ is the adiabatically rearranged density field defined by the requirement that it minimizes the potential energy of the system. The resulting density field is a non-increasing function of z and is independent of time by virtue of the assumed non-diffusive nature of the fluid; however for realistic flows it will be time dependent. We assume that there exists a differentiable, one-to-one, volume-preserving mapping $(x, z) = \Phi_L(x', z', t) = (x^*_L(x', z', t), z^*_L(x', z', t))$ between the original and final states. Thus, a fluid parcel at (x', z', t) in the perturbed state has height $z^*_L(x', z', t)$ in the reference state. Let $\Phi_L^{-1}(x^*, z^*, t) = (x_L(x^*, z^*, t), z_L(x^*, z^*, t))$ be the inverse. There is no restriction to perturbed states with no density overturns. The density of a fluid particle is preserved under the mapping,

$$\bar{\rho}_L(z^*_L(x', z', t)) = \rho(x', z', t). \tag{2.3}$$

Under this mapping

$$\int_{-L}^{L} \int_{-H}^{0} \bar{\rho}_L(z) z \, \mathrm{d}x \, \mathrm{d}z = \int_{-L}^{L} \int_{-H}^{0} \rho(x', z', t) z^*_L(x', z', t) J_L(x', z', t) \, \mathrm{d}x' \, \mathrm{d}z', \tag{2.4}$$

where J_L is the Jacobian of the transformation which, since the mapping is volume preserving, is equal to one. The density field need not be continuous, so the results below are applicable to multi-layered fluids. Differentiability of the mapping does impose some conditions. For example, the perturbed density field cannot have extrema in the interior of the fluid.

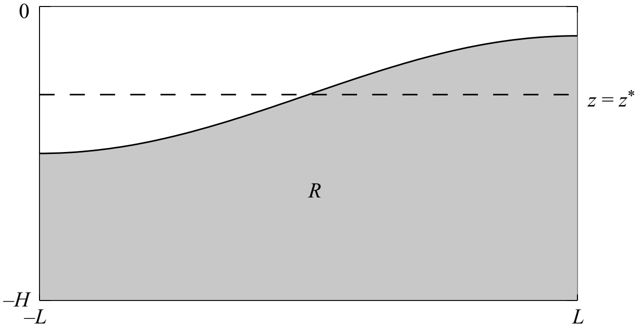

FIGURE 1. The shaded region R is bounded by $x = \pm L$, $z = -H$ and $z = z_L(x^*, z^*)$. Under the mapping Φ_L the upper boundary $z_L(x^*, z^*)$ goes to $z = z^*$ which is the mean height of the surface $z = z_L(x^*, z^*)$. The volumes of fluid under $z = z_L(x^*, z^*)$ and $z = z^*$ are identical.

Using (2.4), (2.1) can be rewritten as

$$\begin{aligned} APE_L &= \int_{-L}^{L}\int_{-H}^{0} g\rho(x,z,t)(z - z^*(x,z,t))\,\mathrm{d}z\,\mathrm{d}x, \\ &= \int_{-L}^{L}\int_{-H}^{0} g\rho(x,z,t)z\,\mathrm{d}z\,\mathrm{d}x - \int_{-L}^{L}\int_{-H}^{0} g\rho(x,z,t)z^*(x,z,t)\,\mathrm{d}z\,\mathrm{d}x, \\ &= PE_p - PE_r, \end{aligned} \tag{2.5}$$

where PE_p is the potential energy of the perturbed state and PE_r is the potential energy of the reference state, called E_p and E_b respectively in Winters *et al.* (1995). The area of the region R in the (x, z)-plane (see figure 1) bounded by $z = -H$ and $z = z_L(x^*, z^*)$ for constant z^* is preserved under the mapping. Hence

$$2L(z^* + H) = \frac{1}{2}\oint_{\partial R} (x, z)\boldsymbol{\cdot}\hat{\boldsymbol{n}}\mathrm{d}s, \tag{2.6}$$

where ∂R is the boundary of R, which easily leads to

$$z^* = \frac{1}{2L}\int_{-L}^{L} z_L \frac{\partial x_L}{\partial x^*}\,\mathrm{d}x^*. \tag{2.7}$$

Since the relaxed state depends on the region in which the sorting is done, so does the available potential energy. This is a particular problem in open systems due to the presence of other features (e.g. eddies, geostrophic currents, internal tides, etc.) in the far field but is well defined in a closed system such as a tank in the laboratory (Scotti *et al.* 2006). A further difficulty with the above expression is that it is a global one. Fortunately the available potential energy density (1.2) can be used which gives information about the spatial distribution of available potential energy (Holliday & McIntyre 1981; Shepherd 1993; Scotti *et al.* 2006; Lamb 2007). In terms of E_a

$$APE_L = \int_{-L}^{L}\int_{-H}^{0} E_a(x,z,t)\,\mathrm{d}z\,\mathrm{d}x, \tag{2.8}$$

where $\bar{\rho}_L(z)$ and $z_L^*(x, z, t)$ are the reference density and reference height of the fluid parcels used in the expression for E_a. The equivalence of the two formulations is proved in the next section.

The equations governing the evolution of the flow are those for an inviscid, non-diffusive fluid in a rotating reference frame

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+\rho 2\boldsymbol{\Omega}\times\boldsymbol{u}=-\nabla p-\rho g\hat{\boldsymbol{k}}, \tag{2.9}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}=0, \tag{2.10}$$

$$\nabla\cdot\boldsymbol{u}=0. \tag{2.11}$$

Here $\boldsymbol{u}=(u,v,w)$ is the velocity vector, $\mathrm{D}/\mathrm{D}t=\partial/\partial t+\boldsymbol{u}\cdot\nabla$ is the material derivative, $\boldsymbol{\Omega}$ is the angular velocity of the Earth, p the pressure, $\hat{\boldsymbol{k}}$ is the unit vector in the vertical direction and g is the gravitational acceleration. From these a conservation law for E_k+E_w,

$$\frac{\partial}{\partial t}(E_k+E_w)+\nabla\cdot((E_k+E_w+\bar{\rho}_L gz+\bar{p}_L(z)+p_w(x,z,t))\boldsymbol{u})=0, \tag{2.12}$$

is easily derived. Here the pressure p has been split into a part in hydrostatic balance with the reference density,

$$\bar{p}_L(z)=\int_z^0\bar{\rho}_L(z)g\,\mathrm{d}z, \tag{2.13}$$

and a remainder $p_w(x,z,t)$, associated with the perturbation. The latter could be further separated into a surface perturbation $p_s(x,t)$, a part in hydrostatic balance with the density perturbution and a non-hydrostatic contribution (Venayagamoorthy & Fringer 2005; Moum *et al.* 2007).

For a wave field in the absence of a barotropic flow the horizontal flux terms $u\bar{\rho}_L gz$ and $u\bar{p}_L$ are linear in the perturbation, uE_w is quadratic and uE_k is cubic. For periodic waves the linear flux terms have zero mean. These terms were ignored by Moum *et al.* (2007) in the context of internal solitary waves; however as shown by Lamb (2007), for internal solitary waves these terms are non-zero and make the dominant contribution to the energy flux. This is easily shown by considering weakly nonlinear solitary waves. Introducing a stream function Ψ such that $(u,w)=(\Psi_z,-\Psi_x)$ with boundary conditions $\Psi=0$ at $z=-H,0$ assuming no barotropic flow, the linear flux terms can be written as

$$\int_{-H}^0 u(\bar{\rho}_L gz+\bar{p}_L)\,\mathrm{d}z=-\int_{-H}^0\Psi\frac{\mathrm{d}\bar{\rho}_L}{\mathrm{d}z}gz\,\mathrm{d}z \tag{2.14}$$

For long, weakly nonlinear waves in the absence of rotation, the stream function satisfies

$$\Psi\frac{\mathrm{d}\bar{\rho}_L}{\mathrm{d}z}g=c^2(\bar{\rho}_L\Psi_z)_z, \tag{2.15}$$

where c is the linear long wave propagation speed, giving

$$\int_{-H}^0 u(\bar{\rho}_L gz+\bar{p}_L)\,\mathrm{d}z=-c^2H\rho_b u_b(x,t)+\frac{c^4}{g}(\rho_b u_b(x,t)-\rho_s u_s(x,t)), \tag{2.16}$$

where the subscripts b and s refer to values along the bottom and surface. Making the Boussinesq approximation the term with the factor c^4/g is absent and ρ_b is replaced by the reference density ρ_0 in the first term, clearly showing this term is generally non-zero.

The pseudoenergy $E_{pseudo} = E_k + E_a$ satisfies the pseudoenergy equation

$$\frac{\partial}{\partial t}(E_k + E_a) + \nabla \cdot (\boldsymbol{u}(E_k + E_a + p_w)) = 0 \tag{2.17}$$

(Holliday & McIntyre 1981; Scotti *et al.* 2006; Lamb 2007), which is simpler than (2.12). There are no flux terms linear in the perturbation: the horizontal flux up_w is quadratic in the perturbation while the other two flux terms are cubic.

3. Comparison of E_w and E_a

In this section we first show that under fairly general conditions

$$\int_{-L}^{L}\int_{-H}^{0} E_w \,\mathrm{d}z\,\mathrm{d}x = \int_{-L}^{L}\int_{-H}^{0} E_a \,\mathrm{d}z\,\mathrm{d}x, \tag{3.1}$$

and then present a simple example to illustrate some key concepts.

A simple argument, first presented by Holliday & McIntyre (1981), shows that the time derivatives of the two integrals above are equal. The essence of their argument is as follows. Because the flow is incompressible, assuming no flow through the lateral boundaries at $x = \pm L$ so that $[-L, L] \times [-H, 0]$ is a material volume,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-L}^{L}\int_{-H}^{0} E_{a,w} \,\mathrm{d}z\,\mathrm{d}x = \int_{-L}^{L}\int_{-H}^{0} \frac{\mathrm{D}E_{a,w}}{\mathrm{D}t} \,\mathrm{d}z\,\mathrm{d}x. \tag{3.2}$$

From (2.17) and (2.12) the material derivatives of E_a and E_w differ by the divergence of a flux vector of the form $\vec{\nabla} \cdot (uh)$. In particular, the flux vector is proportional to u and hence is zero on the boundaries. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-L}^{L}\int_{-H}^{0} E_a \,\mathrm{d}z\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-L}^{L}\int_{-H}^{0} E_w \,\mathrm{d}z\,\mathrm{d}x. \tag{3.3}$$

Thus, if the two integrals are equal initially they remain equal. This result also holds in a laterally unbounded domain assuming that $u \rightarrow 0$ as $x \rightarrow \pm\infty$. For a viscous, non-diffusive fluid $u \rightarrow 0$ and $\rho(x, z, t) \rightarrow \bar{\rho}_L(z)$ as $t \rightarrow \infty$, in which case E_a and E_w both go to zero. Since the integrals of E_a and E_w are then identical in the limit $t \rightarrow \infty$ they must be identical at all times.

We next demonstrate the equality of the two formulations directly. As this approach does not use time derivatives it is valid for a viscous, diffusive fluid. Holliday & McIntyre (1981) proved their equivalence by noting that $\rho(x, z) = \bar{\rho}_L(z - z_L^*(x, z, t))$ and by then expanding in a Taylor series about z. This requires that $\bar{\rho}_L$ be infinitely differentiable and also imposes a limit on the amplitude of the perturbation that can be somewhat restrictive. For example, for the hyperbolic tangent stratification $\bar{\rho}_L = \rho_0(1 - \delta\rho \tanh((z - z_0)/d))$, the Taylor series expansion about $z = z_0$ has a radius of convergence of $\pi d/2 \approx 1.57d$. The following proof has no such limitation and does not require $\rho(x, z, t)$ to be differentiable. Hence is it applicable to layered stratifications.

Consider

$$\int_{-L}^{L}\int_{-H}^{0} E_a \,\mathrm{d}z\,\mathrm{d}x = g\int_{-L}^{L}\int_{-H}^{0}\int_{z}^{z_L^*(x,z,t)} (\bar{\rho}_L(s) - \rho(x, z, t)) \,\mathrm{d}s\,\mathrm{d}z\,\mathrm{d}x. \tag{3.4}$$

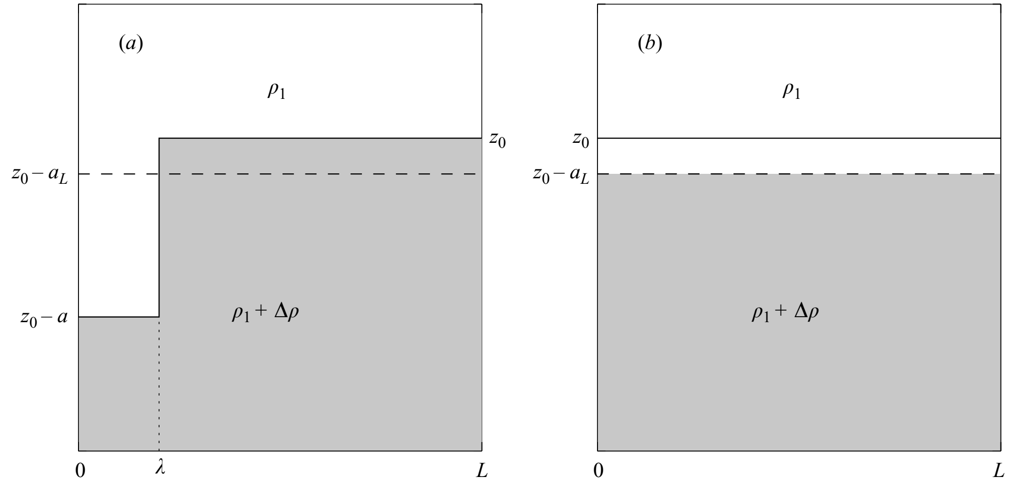

FIGURE 2. Example two-layer stratification. (*a*) Perturbed stratification. (*b*) Rest state.

Integrating the integral in z of the piece involving $\bar{\rho}_L(s)$ by parts, using $z_L^*(x, z, t) = z$ along $z = -H, 0$, and (2.3) and (2.4) gives

$$\int_{-L}^{L}\int_{-H}^{0} E_a \,\mathrm{d}z\,\mathrm{d}x = g\int_{-L}^{L}\int_{-H}^{0}\left\{\rho z - z\rho\frac{\partial z_L^*}{\partial z}\right\}\mathrm{d}z\,\mathrm{d}x. \tag{3.5}$$

Using the transformation $(x^*, z^*) = \Phi_L(x, z)$ in the second term in the integrand gives

$$\int_{-L}^{L}\int_{-H}^{0} E_a \,\mathrm{d}z\,\mathrm{d}x = g\int_{-L}^{L}\int_{-H}^{0}\rho z\,\mathrm{d}x\,\mathrm{d}z - g\int_{-L}^{L}\int_{-H}^{0}\bar{\rho}_L(z^*)z_L(x^*, z^*)\frac{\partial x_L}{\partial x^*}\,\mathrm{d}z^*\,\mathrm{d}x^*. \tag{3.6}$$

Using (2.7) then gives the result.

3.1. *A simple example*

To illustrate some key results consider the perturbed two-layer stratification shown in figure 2(*a*) (for other examples see Hebert (1988)). The upper layer has density ρ_1 and the lower layer density $\rho_2 = \rho_1 + \Delta\rho$. The perturbed interface takes the form of a square well, with the interface at

$$z = \begin{cases} z_0 - a & \text{if } 0 < x < \lambda \\ z_0 & \text{if } \lambda < x < L. \end{cases} \tag{3.7}$$

For this closed system, the interface in the reference density, shown in figure 2(*b*), is at $z = z_0 - a_L$ where

$$\lambda a = L a_L. \tag{3.8}$$

Consider the region $0 < x < \lambda$. For $z > z_0 - a_L$ a fluid particle has rest height above $z_o - a_L$ so $\bar{\rho}(s) - \rho = 0$ for $z < s < z^*$, hence $E_a = 0$ if $z > z_0 - a_L$. Similarly E_a is zero for $z < z_0 - a$. For $z_0 - a < z < z_0 - a_L$, fluid at height z has rest height above $z_0 - a_L$, i.e. $z^* > z_0 - a_L$. In this case

$$\bar{\rho}_L(s) - \rho = \begin{cases} 0 & \text{if } s > z_0 - a_L \\ \Delta\rho & \text{if } z < s < z_0 - a_L, \end{cases} \tag{3.9}$$

so

$$g\int_{z}^{z^*} (\bar{\rho}_L - \rho)\,\mathrm{d}s = g\int_{z}^{z_0-a_L} (\bar{\rho}_L - \rho)\,\mathrm{d}s = -g\Delta\rho(z-(z_0-a_L)). \tag{3.10}$$

Summarizing, for $0 < x < \lambda$,

$$E_a = \begin{cases} 0 & \text{if } z > z_0 - a_L \\ -g\Delta\rho(z-(z_0-a_L)) & \text{if } z_0 - a < z < z_0 - a_L \\ 0 & \text{if } z < z_0 - a. \end{cases} \tag{3.11}$$

Similar considerations show that for $\lambda < x < L$,

$$E_a = \begin{cases} 0 & \text{if } z > z_0 \\ g\Delta\rho(z-(z_0-a_L)) & \text{if } z_0 - a_L < z < z_0 \\ 0 & \text{if } z < z_0 - a_L. \end{cases} \tag{3.12}$$

Note that E_a is always positive. We now integrate E_a to find the available potential energy. The contribution from $0 < x < \lambda$ is

$$-\lambda g\Delta\rho\int_{z_0-a}^{z_0-a_L} (z-(z_0-a_L))\,\mathrm{d}z = g\frac{\Delta\rho}{2}\lambda(a-a_L)^2, \tag{3.13}$$

while the contribution from $\lambda < x < L$ is

$$(L-\lambda)\int_{z_0-a_L}^{z_0} (z-(z_0-a_L))\,\mathrm{d}z = (L-\lambda)a_L^2. \tag{3.14}$$

Summing gives

$$APE_L = g\frac{\Delta\rho}{2}\{\lambda(a-a_L)^2 + (L-\lambda)a_L^2\} = g\frac{\Delta\rho}{2}\{\lambda a^2 - La_L^2\}, \tag{3.15}$$

since $\lambda a = La_L$.

Now consider the integral of E_w. For $0 < x < \lambda$,

$$E_w = \begin{cases} 0 & \text{if } z > z_0 - a_L \\ -g\Delta\rho z & \text{if } z_0 - a < z < z_0 - a_L \\ 0 & \text{if } z < z_0 - a, \end{cases} \tag{3.16}$$

while for $\lambda < x < L$,

$$E_a = \begin{cases} 0 & \text{if } z > z_0 \\ g\Delta\rho z & \text{if } z_0 - a_L < z < z_0 \\ 0 & \text{if } z < z_0 - a_L. \end{cases} \tag{3.17}$$

E_w will take on both positive and negative values unless $z_0 - a_L = 0$. Integrating E_w, the contribution from $0 < x < \lambda$ is

$$\lambda\int_{z_0-a}^{z_0-a_L} -\Delta\rho g z\,\mathrm{d}z = g\frac{\Delta\rho}{2}\{\lambda\big(-2z_0(a-a_L) + a^2 - a_L^2\big)\}, \tag{3.18}$$

while the contribution from $\lambda < x < L$ is

$$(L-\lambda)\int_{z_0-a_L}^{z_0} \Delta\rho g z\,\mathrm{d}z = g\frac{\Delta\rho}{2}\{(L-\lambda)\big(2z_0a_L - a_L^2\big)\}. \tag{3.19}$$

Summing the two, using $\lambda a = La_L$, recovers (3.15).

Next, consider the limit $L \to \infty$. As $L \to \infty$, $a_L \to 0$ while $La_L = \lambda a$ remains constant. Hence $La_L^2 \to 0$ and

$$APE_L \to APE_\infty = \lambda g \frac{\Delta\rho}{2} a^2. \tag{3.20}$$

This is the limiting contribution of the integral of E_a in $0 < x < \lambda$. The region $x > \lambda$ makes no contribution in the limit $L \to \infty$. When integrating E_w, on the other hand, as $L \to \infty$ the contribution from $0 < x < \lambda$ is

$$\lambda g \frac{\Delta\rho}{2}(-2z_0 a + a^2), \tag{3.21}$$

while the region $x > \lambda$ contributes

$$\lambda g \frac{\Delta\rho}{2}(2z_0 a). \tag{3.22}$$

The sum recovers (3.20); however the region outside the perturbation now makes a finite contribution in the limit $L \to \infty$. The terms which are linear in a depend on the location of the zero height, that is they depend on the choice of coordinate systems. They disappear if the coordinate system is chosen such that $z_0 = 0$.

Suppose one takes the limit $L \to \infty$ before determining the reference density $\bar{\rho}(z)$. Doing so gives $\bar{\rho}(z) = \rho_1$ for $z > z_0$ and $\rho_1 + \Delta\rho$ for $z < z_0$. The available potential energy density is now

$$E_a = \begin{cases} -g\Delta\rho(z - z_0) & \text{if } 0 < x < \lambda \quad \text{and} \quad z_0 - a < z < z_0 \\ 0 & \text{otherwise,} \end{cases} \tag{3.23}$$

the integral of which recovers (3.20). Integrating the limiting form of E_w, in contrast, gives

$$\int_{-\lambda}^{\lambda} \int_{-H}^{0} (\rho - \bar{\rho}) g z \, \mathrm{d}x \, \mathrm{d}z = -\lambda g \Delta\rho \int_{z_0 - a}^{z_0} z \, \mathrm{d}z = \lambda g \frac{\Delta\rho}{2}(a^2 - 2az_0), \tag{3.24}$$

which is incorrect. Furthermore the results depend on the coordinate system (i.e. the value of z_o). Thus, to find APE_∞ using E_a one can use the far-field density as the reference density. This is not possible when using E_w (Hebert 1988).

The limiting APE can be written as

$$APE_\infty = g \frac{\Delta\rho}{2} \int_0^\infty \eta^2(x) \, \mathrm{d}x, \tag{3.25}$$

where

$$\eta = \begin{cases} -a & \text{if } 0 < x < \lambda \\ 0 & \text{if } x > \lambda \end{cases} \tag{3.26}$$

is the interface displacement from its far-field height. This is the expression for the APE of an internal solitary wave in a two-layer fluid used by Bogucki & Garrett (1993).

4. The available potential energy of an isolated feature in an infinite domain

We now demonstrate that the far-field density $\bar{\rho}(z)$ can be used as the reference density when calculating the available potential energy of a general isolated feature in an infinite domain by integrating E_a.

Consider an isolated disturbance confined to lie in the horizontal domain $D_\lambda = [-\lambda, \lambda]$ so that $\rho(x, z) = \bar{\rho}(z)$ for $|x| > \lambda$. Consider a domain $D_L = [-L, L]$ where $L > \lambda$, where ultimately we will let $L \to \infty$. Time dependence is suppressed as we do not consider temporal evolution in this section. We want to show that

$$\lim_{L\to\infty} APE_L = \int_{-\lambda}^{\lambda} \int_{-H}^{0} \lim_{L\to\infty} E_a \, \mathrm{d}z \, \mathrm{d}x. \tag{4.1}$$

Let $q(x, z)$ be a differentiable scalar field with non-zero gradient with the property that fluid parcels with the same value of q have the same density. We can take q as $\lim_{L\to\infty} z_L^*(x, z)$. Under $(x, z) \to \Phi_L(x, z)$, $q \to \bar{q}_L(z^*)$. Let $q = \bar{q}(z)$ for $|x| > \lambda$ and denote the inverses of $\bar{q}_L(z)$ and $\bar{q}(z)$ by $\bar{z}_L(q)$ and $\bar{z}(q)$. It is easy to verify that for $x \in (-\lambda, \lambda)$

$$\bar{z}_L(q) = \bar{z}(q) + \frac{\lambda}{L}[\bar{z}_\lambda(q) - \bar{z}(q)]. \tag{4.2}$$

Using this and the fact that $\rho(x, z) = \bar{\rho}(z)$ for $|x| > \lambda$, expression (2.5) for APE_L can be written as

$$\begin{aligned} APE_L = &\int_{-\lambda}^{\lambda} \int_{-H}^{0} \rho(x, z) g \Big\{ z - \bar{z}(q(x, z)) - \frac{\lambda}{L} \Big[\bar{z}_\lambda(q(x, z)) - \bar{z}(q(x, z)) \Big] \Big\} \, \mathrm{d}z \, \mathrm{d}x \\ &-2\lambda \left(1 - \frac{\lambda}{L} \right) \int_{-H}^{0} g \bar{\rho}(z) [\bar{z}_\lambda(\bar{q}(z)) - z] \, \mathrm{d}z. \end{aligned} \tag{4.3}$$

Letting $L \to \infty$ gives

$$APE_\infty = \int_{-\lambda}^{\lambda} \int_{-H}^{0} \{ \rho(x, z) g [z - \bar{z}(q(x, z))] - \bar{\rho}(z) g [\bar{z}_\lambda(\bar{q}(z)) - z] \} \, \mathrm{d}z \, \mathrm{d}x. \tag{4.4}$$

If $\bar{\rho}(z)$ is invertible then one can choose $q = \rho$, in which case this expression could be used to compute the available potential energy given the density field $\rho(x, z)$ and far-field density $\bar{\rho}(z)$. We now need to show that, using $\bar{\rho}(z)$ as the reference density for E_a, this is equal to

$$APE = \int_{-\lambda}^{\lambda} \int_{-H}^{0} E_a \, \mathrm{d}z \, \mathrm{d}x, \tag{4.5}$$

since $E_a = 0$ for $|x| > \lambda$. Using the definition of E_a

$$APE = \int_{-\lambda}^{\lambda} \int_{-H}^{0} \left\{ \rho(x, z) g [z - \bar{z}(q(x, z))] + g \int_{z}^{\bar{z}(q(x,z))} \bar{\rho}(s) \, \mathrm{d}s \right\} \mathrm{d}z \, \mathrm{d}x, \tag{4.6}$$

hence we need to show that

$$\int_{-\lambda}^{\lambda} \int_{-H}^{0} \bar{\rho}(z) g \Big[z - \bar{z}_\lambda(\bar{q}(z)) \Big] \, \mathrm{d}z = \int_{-\lambda}^{\lambda} \int_{-H}^{0} g \int_{z}^{\bar{z}(q(x,z))} \bar{\rho}(s) \, \mathrm{d}s \, \mathrm{d}z. \tag{4.7}$$

Integrating by parts, using $\bar{z}(q(x, 0)) = 0$ and $\bar{z}(q(x, -H)) = -H$,

$$\int_{-H}^{0} g \int_{z}^{\bar{z}(q(x,z))} \bar{\rho}(s) \, \mathrm{d}s \, \mathrm{d}z = g \int_{-H}^{0} \left\{ \bar{\rho}(z) z - z \rho(x, z) \frac{\partial \bar{z}}{\partial q}(q(x, z)) \frac{\partial q}{\partial z}(x, z) \right\} \mathrm{d}z, \tag{4.8}$$

thus the problem reduces to showing that

$$\int_{-\lambda}^{\lambda} \int_{-H}^{0} z \rho(x, z) \frac{\partial \bar{z}}{\partial q} \big(q(x, z) \big) \frac{\partial q}{\partial z}(x, z) \, \mathrm{d}z = \int_{-\lambda}^{\lambda} \int_{-H}^{0} \bar{\rho}(z) \bar{z}_\lambda(\bar{q}(z)) \, \mathrm{d}x \, \mathrm{d}z. \tag{4.9}$$

This is straightforward if $q(x, z)$ is invertible in z with inverse $z(x, q)$, in which case the change of variables $z = z(x, \bar{q}(s))$ quickly leads to the result. More generally, consider the left-hand side of (4.9), which we denote by I. Changing variables to $(x^*, z^*) = \Phi_\lambda(x, z)$ the integral on the left-hand side can be written as

$$I = g \int_{-\lambda}^{\lambda} \int_{-H}^{0} \bar{\rho}_\lambda(z^*) \frac{\partial \bar{z}}{\partial q}(\bar{q}_\lambda(z^*)) \frac{\partial \bar{q}_\lambda}{\partial z}(z^*) z_\lambda(x^*, z^*) \frac{\partial z_\lambda^*}{\partial z}\left(\Phi_L^{-1}(x^*, z^*)\right) \mathrm{d}z^* \, \mathrm{d}x^*. \tag{4.10}$$

Now

$$\frac{\partial z_\lambda^*}{\partial z}\left(\Phi_L^{-1}(x^*, z^*)\right) = \frac{\partial x_\lambda}{\partial x^*}(x^*, z^*), \tag{4.11}$$

and

$$\int_{-\lambda}^{\lambda} z_\lambda(x^*, z^*) \frac{\partial x_\lambda}{\partial x^*}(x^*, z^*) \, \mathrm{d}x^* = \bar{z}_\lambda(\bar{q}_\lambda(z^*)) \tag{4.12}$$

(see 2.7), so the integral can be written as

$$I = g \int_{-\lambda}^{\lambda} \int_{-H}^{0} \bar{\rho}_\lambda(z^*) \frac{\partial \bar{z}}{\partial q}(\bar{q}_\lambda(z^*)) \frac{\partial \bar{q}_\lambda}{\partial z}(z^*) \bar{z}_\lambda(\bar{q}_\lambda(z^*)) \, \mathrm{d}z^* \, \mathrm{d}x^*. \tag{4.13}$$

We now introduce a new mapping $f(z)$ via $\bar{q}(z) = \bar{q}_\lambda(f(z))$. Changing the variable of integration from z^* to z where $z^* = f(z)$ then leads to

$$I = g \int_{-\lambda}^{\lambda} \int_{-H}^{0} \bar{\rho}(z) \bar{z}_\lambda(\bar{q}(z)) \, \mathrm{d}z \, \mathrm{d}x^*, \tag{4.14}$$

since $\bar{\rho}(z) = \bar{\rho}_\lambda(f(z))$, proving the desired result.

5. Summary

We have shown that the available potential energy can be calculated by either integrating the perturbation potential energy E_w or the available potential energy density E_a. This generalizes the result of Holliday & McIntyre (1981) which was restricted to smooth stratifications for which the density perturbation could be written in terms of a convergent Taylor series expansion of the reference stratification. Use of the available potential energy density E_a has several advantages. The pseudoenergy density $E_k + E_a$ satisfies a relatively simple conservation law. E_a is positive definite, independent of the coordinate system and recovers the linear potential energy density in the small-amplitude limit. It also simplifies the calculation of the available potential energy of an isolated feature in an unbounded domain because the far-field density can be used as the reference density. This is a useful property, for example, when comparing the available potential energies in internal solitary waves before and after reflection from a sloping bottom. Using E_a there is no need to sort the density fields to find a reference density in order to calculate the APE. Integrating E_w to find the APE, on the other hand, requires sorting the density to find the reference density in a finite domain and then finding the limiting value of the APE as the domain length goes to infinity.

Use of E_a could also simplify estimates of the APE of an isolated wave in observations in which the integral of E_w has been used, necessitating sorting the density field over a large domain to obtain the reference density (Klymak *et al.* 2006; Moum *et al.* 2007).

The above results can be readily extended to three dimensions and to flows with a free surface. The latter is illustrated by the example considered in §3 for which the interface could be the air-water interface. In this example the rigid lid could also be at z_0 demonstrating that the ideas presented here are also applicable to perturbations that include density anomalies, e.g. warm-core eddies or internal solitary waves with a trapped core.

Finally, it should be noted that it is not clear that the APE in an unbounded domain is the most physically relevent APE. For example, when an isolated wave in an unbounded domain breaks, resulting in mixing, it does so in a finite subregion. What is the 'ideal' size of the domain in which to calculate the APE? There are also open problems associated with estimating the APE of non-isolated features. These and other interesting questions about APE remain for future consideration.

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